Maximising and Minimising

Tip (Method)
1. Write the quantity to be optimised as a function of *one* variable. (If two variables appear, eliminate one using the constraint given in the question.)
2. Differentiate, set the derivative to zero, solve.
3. Justify maximum or minimum (second derivative, or gradient either side).
4. Answer the actual question — often the value of the quantity, not of x.

Example (Edexcel C2)

The volume V cm^3 of a box of height x cm is given by

$$V = 4x(5-x)^2, \qquad 0 < x < 5.$$

1. Find $\dfrac{dV}{dx}$.
2. Hence find the maximum volume of the box.
3. Use calculus to justify that your answer is a maximum.

1. $V = 4x(25-10x+x^2) = 100x - 40x^2 + 4x^3$, *so* $\dfrac{dV}{dx} = 100 - 80x + 12x^2$.

2. $4(3x-5)(x-5) = 0 \implies x = \frac{5}{3}$ *(rejecting $x = 5$, outside the domain).*

$V = 4 \cdot \frac{5}{3}\left(5 - \frac{5}{3}\right)^2 = \frac{20}{3} \cdot \frac{100}{9} = \frac{2000}{27} \approx 74.1\,\text{cm}^3$

3. $\dfrac{d^2V}{dx^2} = -80 + 24x$; *at* $x = \frac{5}{3}$: $-40 < 0$: *maximum.*

Example

A farmer has 120 m of fencing for a rectangular pen against a long straight wall; the wall forms one side, fencing the other three. Find the dimensions giving the greatest area, and that area.

Width x (two sides), length $120 - 2x$*:* $A = x(120 - 2x) = 120x - 2x^2$.

$\dfrac{\mathrm{d}A}{\mathrm{d}x} = 120 - 4x = 0 \implies x = 30$; $\dfrac{\mathrm{d}^2A}{\mathrm{d}x^2} = -4 < 0$*: maximum.*

$30\,\mathrm{m}$ *by* $60\,\mathrm{m}$, *area* $1800\,\mathrm{m}^2$.

Textbook Exercises: SPS Course 6.1, Exercise 5 (optimisation questions)

Optimisation with Constraints

Example (Edexcel C2)

A solid glass cylinder has volume 75π cm^3. Polishing costs £2 per cm^2 for the curved surface and £3 per cm^2 for the circular top and base. The radius is r cm.

1. Show that the cost of polishing is $C = 6\pi r^2 + \dfrac{300\pi}{r}$.
2. Use calculus to find the minimum cost, to the nearest pound.
3. Justify that your answer is a minimum.

1. $\pi r^2 h = 75\pi \implies h = \dfrac{75}{r^2}$.

$C = 2(2\pi r h) + 3(2\pi r^2) = 4\pi r \cdot \dfrac{75}{r^2} + 6\pi r^2 = 6\pi r^2 + \dfrac{300\pi}{r}$

2. $\dfrac{\mathrm{d}C}{\mathrm{d}r} = 12\pi r - \dfrac{300\pi}{r^2} = 0 \implies r^3 = 25 \implies r = 25^{1/3}$

$C = 6\pi \cdot 25^{2/3} + 300\pi \cdot 25^{-1/3} = 161.2\ldots + 322.3\ldots = £483$ *(nearest pound).*

3. $\dfrac{\mathrm{d}^2C}{\mathrm{d}r^2} = 12\pi + \dfrac{600\pi}{r^3} > 0$ *for* $r > 0$*: minimum.*

Example

An open-topped box is made from a square sheet of card of side 24 cm by cutting a square of side x cm from each corner and folding up the sides.

1. Show that the volume of the box is $V = x(24-2x)^2$.
2. Find the value of x that maximises the volume, and the maximum volume.

1. *Base* $(24-2x)\times(24-2x)$, *height* x.
2. $V = 576x - 96x^2 + 4x^3$: $\dfrac{dV}{dx} = 576 - 192x + 12x^2 = 12(x-4)(x-12) = 0$

   $x = 4$ *(rejecting* $x = 12$*, which gives no base).* $\dfrac{d^2V}{dx^2} = 24x - 192 < 0$ *at* $x = 4$*: maximum.*

   $V = 4\times 16^2 = 1024\,\text{cm}^3$

Exercise. A closed cylindrical can must hold $330\,\text{cm}^3$. Find, in terms of π, the radius that minimises the surface area, and show that the optimal can has height equal to its diameter.

Textbook Exercises: SPS Course 6.1, Exercise 6 (Miscellaneous problems)